

Gaussian approximations of multiple integrals

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Abstract

Fix $k \geq 1$, and let $\mathbf{I}(l)$, $l \geq 1$, be a sequence of k -dimensional vectors of multiple Wiener-Itô integrals with respect to a general Gaussian process. We establish necessary and sufficient conditions to have that, as $l \rightarrow +\infty$, the law of $\mathbf{I}(l)$ is asymptotically close (for example, in the sense of Prokhorov's distance) to the law of a k -dimensional Gaussian vector having the same covariance matrix as $\mathbf{I}(l)$. The main feature of our results is that they require minimal assumptions (basically, boundedness of variances) on the asymptotic behaviour of the variances and covariances of the elements of $\mathbf{I}(l)$. In particular, we will not assume that the covariance matrix of $\mathbf{I}(l)$ is convergent. This generalizes the results proved in Nualart and Peccati (2005), Peccati and Tudor (2005) and Nualart and Ortiz-Latorre (2007). As shown in Marinucci and Peccati (2007b), the criteria established in this paper are crucial in the study of the high-frequency behaviour of stationary fields defined on homogeneous spaces.

Keywords and Phrases: Central Limit Theorems; Gaussian Approximations; High-frequency CLTs; Multiple Wiener-Itô Integrals; Spectral Analysis; Spherical fields.

MSC: 60B15; 60F05; 60G60

1 Introduction

Let $\mathbf{U}(l) = (U_1(l), \dots, U_k(l))$, $l \geq 1$, be a sequence of centered random observations (not necessarily independent) with values in \mathbb{R}^k . Suppose that the application $l \mapsto \mathbb{E}U_i(l)^2$ is bounded for every i , and also that the sequence of covariances $c_l(i, j) = \mathbb{E}U_i(l)U_j(l)$ does not converge as $l \rightarrow +\infty$ (that is, for some fixed $i \neq j$, the limit $\lim_{l \rightarrow \infty} c_l(i, j)$ does not exist). Then, a natural question is the following: *is it possible to establish criteria ensuring that, for large l , the law of $\mathbf{U}(l)$ is close (in the sense of some distance between probability measures) to the law of a Gaussian vector $\mathbf{N}(l) = (N_1(l), \dots, N_k(l))$ such that $\mathbb{E}N_i(l)N_j(l) = \mathbb{E}U_i(l)U_j(l) = c_l(i, j)$?* Note that the question is not trivial, since the asymptotic irregularity of the covariance matrix $c_l(\cdot, \cdot)$ may in general prevent $\mathbf{U}(l)$ from converging in law toward a k -dimensional Gaussian distribution.

In this paper, we shall provide an exhaustive answer to the problem above in the special case where the sequence $\mathbf{U}(l)$ has the form

$$\mathbf{U}(l) = \mathbf{I}(l) = \left(I_{d_1} \left(f_l^{(1)} \right), \dots, I_{d_k} \left(f_l^{(k)} \right) \right), \quad l \geq 1, \quad (1)$$

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where the integers $d_1, \dots, d_k \geq 1$ do not depend on l , I_{d_j} indicates a multiple stochastic integral of order d_j (with respect to some isonormal Gaussian process X over a Hilbert space \mathfrak{H} – see Section 2 below for definitions), and each $f_l^{(j)} \in \mathfrak{H}^{\odot d_j}$, $j = 1, \dots, k$, is a symmetric kernel. In particular, we shall prove that, whenever the elements of the vectors $\mathbf{I}(l)$ have bounded variances (and without any further requirements on the covariance matrix of $\mathbf{I}(l)$), the following three conditions are equivalent as $l \rightarrow +\infty$:

- (i) $\gamma(\mathcal{L}(\mathbf{I}(l)), \mathcal{L}(\mathbf{N}(l))) \rightarrow 0$, where $\mathcal{L}(\cdot)$ indicates the law of a given random vector, $\mathbf{N}(l)$ is a Gaussian vector having the same covariance matrix as $\mathbf{I}(l)$, and γ is some appropriate metric on the space of probability measures on \mathbb{R}^k ;
- (ii) For every $j = 1, \dots, k$, $\mathbb{E} \left(I_{d_j} \left(f_l^{(j)} \right)^4 \right) - 3 \mathbb{E} \left(I_{d_j} \left(f_l^{(j)} \right)^2 \right)^2 \rightarrow 0$;
- (iii) For every $j = 1, \dots, k$ and every $p = 1, \dots, d_j - 1$, the sequence of contractions (to be formally defined in Section 2) $f_l^{(j)} \otimes_p f_l^{(j)}$, $l \geq 1$, is such that

$$f_l^{(j)} \otimes_p f_l^{(j)} \rightarrow 0 \quad \text{in } \mathfrak{H}^{\odot 2(d_j-p)}. \quad (2)$$

Some other conditions, involving for instance Malliavin operators, are derived in the subsequent sections. As discussed in Section 5, our results are motivated by the derivation of high-frequency Gaussian approximations of stationary fields defined on homogeneous spaces – a problem tackled in [9] and [10].

Note that the results of this paper are a generalization of the following theorem, which combines results proved in [13], [14] and [15].

Theorem 0. *Suppose that the vector $\mathbf{I}(l)$ in (1) is such that, as $l \rightarrow +\infty$,*

$$\mathbb{E} I_{d_i} \left(f_l^{(i)} \right) I_{d_j} \left(f_l^{(j)} \right) \rightarrow \mathbf{C}(i, j), \quad 1 \leq i, j \leq k,$$

where $\mathbf{C} = \{\mathbf{C}(i, j)\}$ is some positive definite matrix. Then, the following four conditions are equivalent, as $l \rightarrow +\infty$:

1. $\mathbf{I}(l) \xrightarrow{Law} \mathbf{N}(0, \mathbf{C})$, where $\mathbf{N}(0, \mathbf{C})$ is a k -dimensional centered Gaussian vector with covariance matrix \mathbf{C} ;
2. Relation (2) takes place for every $j = 1, \dots, k$ and every $p = 1, \dots, d_j - 1$;
3. For every $j = 1, \dots, k$, $\mathbb{E} \left(I_{d_j} \left(f_l^{(j)} \right)^4 \right) \rightarrow 3\mathbf{C}(j, j)^2$;
4. For every $j = 1, \dots, k$, $\left\| D \left[I_{d_j} \left(f_l^{(j)} \right) \right] \right\|_{\mathfrak{H}}^2 \rightarrow d_j$ in L^2 , where $D \left[I_{d_j} \left(f_l^{(j)} \right) \right]$ denotes the Malliavin derivative of $I_{d_j} \left(f_l^{(j)} \right)$ (see the next section).

The equivalence of Points 1.-3. in the case $k = 1$ has been first proved in [14] by means of the Dambis-Dubins-Schwarz (DDS) Theorem (see [16, Ch. V]), whereas the proof in the case

$k \geq 2$ has been achieved (by similar techniques) in [15]; the fact that Point 4. is also necessary and sufficient for the CLT at Point 1. has been recently proved in [13], by means of a Malliavin calculus approach. For some applications of Theorem 0 (in quite different frameworks), see e.g. [2], [3], [5], [9] or [11].

The techniques we use to achieve our main results are once again the DDS Theorem, combined with Burkholder-Davis-Gundy inequalities and some results (taken from [4, Section 11.7]) concerning ‘uniformities’ over classes of probability measures.

The paper is organized as follows. In Section 2 we discuss some preliminary notions concerning Gaussian fields, multiple integrals and metrics on probabilities. Section 3 contains the statements of the main results of the paper. The proof of Theorem 1 (one of the crucial results of this note) is achieved in Section 4. Section 5 is devoted to applications.

2 Preliminaries

We present a brief review of the main notions and results that are needed in the subsequent sections. The reader is referred to [6] or [12, Ch. 1] for any unexplained definition.

Hilbert spaces. In what follows, the symbol \mathfrak{H} indicates a real separable Hilbert space, with inner product $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$ and norm $\|\cdot\|_{\mathfrak{H}}$. For every $d \geq 2$, we denote by $\mathfrak{H}^{\otimes 2}$ and $\mathfrak{H}^{\odot 2}$, respectively, the n th *tensor product* of \mathfrak{H} and the n th *symmetric tensor product* of \mathfrak{H} . We also write $\mathfrak{H}^{\otimes 1} = \mathfrak{H}^{\odot 1} = \mathfrak{H}$.

Isonormal Gaussian processes. We write $X = \{X(h) : h \in \mathfrak{H}\}$ to indicate an *isonormal Gaussian process* over \mathfrak{H} . This means that X is a collection of real-valued, centered and (jointly) Gaussian random variables indexed by the elements of \mathfrak{H} , defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and such that, for every $h, h' \in \mathfrak{H}$,

$$\mathbb{E}[X(h)X(h')] = \langle h, h' \rangle_{\mathfrak{H}}.$$

We denote by $L^2(X)$ the (Hilbert) space of the real-valued and square-integrable functionals of X .

Isometry, chaoses and multiple integrals. For every $d \geq 1$ we will denote by I_d the isometry between $\mathfrak{H}^{\odot d}$ equipped with the norm $\sqrt{d!} \|\cdot\|_{\mathfrak{H}^{\odot d}}$ and the d th Wiener chaos of X . In the particular case where $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$, (A, \mathcal{A}) is a measurable space, and μ is a σ -finite and non-atomic measure, then $\mathfrak{H}^{\odot d} = L_s^2(A^d, \mathcal{A}^{\otimes d}, \mu^{\otimes d})$ is the space of symmetric and square integrable functions on A^d and for every $f \in \mathfrak{H}^{\odot d}$, $I_d(f)$ is the *multiple Wiener-Itô integral* (of order d) of f with respect to X , as defined e.g. in [12, Ch. 1]. It is well-known that a random variable of the type $I_d(f)$, where $d \geq 2$ and $f \neq 0$, *cannot be Gaussian*. Moreover, every $F \in L^2(X)$ admits a unique *Wiener chaotic decomposition* of the type $F = \mathbb{E}(F) + \sum_{d=1}^{\infty} I_d(f_d)$, where $f_d \in \mathfrak{H}^{\odot d}$, $d \geq 1$, and the convergence of the series is in $L^2(X)$.

Malliavin derivatives. We will use Malliavin derivatives in Section 3, where we generalize some of the results proved in [13]. The class \mathcal{S} of *smooth* random variables is defined as the collection of all functionals of the type

$$F = f(X(h_1), \dots, X(h_m)), \tag{3}$$

where $h_1, \dots, h_m \in \mathfrak{H}$ and f is bounded and has bounded derivatives of all order. The operator D , called the *Malliavin derivative operator*, is defined on \mathcal{S} by the relation

$$DF = \sum_{i=1}^M \frac{\partial}{\partial x_i} f(h_1, \dots, h_m) h_i,$$

where F has the form (3). Note that DF is an element of $L^2(\Omega; \mathfrak{H})$. As usual, we define the domain of D , noted $\mathbb{D}^{1,2}$, to be the closure of \mathcal{S} with respect to the norm $\|F\|_{1,2} \triangleq \mathbb{E}(F^2) + \mathbb{E}\|DF\|_{\mathfrak{H}}^2$. When $F \in \mathbb{D}^{1,2}$, we may sometimes write $DF = D[F]$, depending on the notational convenience. Note that any finite sum of multiple Wiener-Itô integrals is an element of $\mathbb{D}^{1,2}$.

Contractions. Let $\{e_k : k \geq 1\}$ be a complete orthonormal system of \mathfrak{H} . For any fixed $f \in \mathfrak{H}^{\odot n}$, $g \in \mathfrak{H}^{\odot m}$ and $p \in \{0, \dots, n \wedge m\}$, we define the p th *contraction* of f and g to be the element of $\mathfrak{H}^{\odot n+m-2p}$ given by

$$f \otimes_p g = \sum_{i_1, \dots, i_p=1}^{\infty} \langle f, e_{i_1} \otimes \dots \otimes e_{i_p} \rangle_{\mathfrak{H}^{\otimes p}} \otimes \langle g, e_{i_1} \otimes \dots \otimes e_{i_p} \rangle_{\mathfrak{H}^{\otimes p}}.$$

We stress that $f \otimes_p g$ need not be an element of $\mathfrak{H}^{\odot n+m-2p}$. We denote by $f \widetilde{\otimes}_p g$ the symmetrization of $f \otimes_p g$. Note that $f \otimes_0 g$ is just the tensor product $f \otimes g$ of f and g . If $n = m$, then $f \otimes_n g = \langle f, g \rangle_{\mathfrak{H}^{\otimes n}}$.

Metrics on probabilities. For $k \geq 1$ we define $\mathbf{P}(\mathbb{R}^k)$ to be the class of all probability measures on \mathbb{R}^k . Given a metric $\gamma(\cdot, \cdot)$ on $\mathbf{P}(\mathbb{R}^k)$, we say that γ *metrizes the weak convergence on $\mathbf{P}(\mathbb{R}^k)$* whenever the following double implication holds for every $Q \in \mathbf{P}(\mathbb{R}^k)$ and every $\{Q_l : l \geq 1\} \subset \mathbf{P}(\mathbb{R}^k)$ (as $l \rightarrow +\infty$): $\gamma(Q_l, Q) \rightarrow 0$ if, and only if, Q_l converges weakly to Q . Some examples of metrizing γ are the *Prokhorov metric* (usually noted ρ) or the *Fortet-Mounier metric* (usually noted β). Recall that

$$\rho(P, Q) = \inf\{\epsilon > 0 : P(A) \leq Q(A^\epsilon) + \epsilon, \text{ for every Borel set } A \subset \mathbb{R}^k\} \quad (4)$$

where $A^\epsilon = \{x : \|x - y\| < \epsilon \text{ for some } y \in A\}$, and $\|\cdot\|$ is the Euclidean norm. Also,

$$\beta(P, Q) = \sup \left\{ \left| \int f d(P - Q) \right| : \|f\|_{BL} \leq 1 \right\}, \quad (5)$$

where $\|\cdot\|_{BL} = \|\cdot\|_L + \|\cdot\|_\infty$, and $\|\cdot\|_L$ is the usual Lipschitz seminorm (see [4, p. 394] for further details). The fact that we focus on the Prokhorov and the Fortet-Mounier metric is due to the following fact, proved in [4, Th. 11.7.1]. *For any two sequences $\{P_l\}, \{Q_l\} \subset \mathbf{P}(\mathbb{R}^k)$, the following three conditions (A)–(C) are equivalent: (A) $\lim_{l \rightarrow +\infty} \beta(P_l, Q_l) = 0$; (B) $\lim_{l \rightarrow +\infty} \rho(P_l, Q_l) = 0$; (C) on some auxiliary probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$, there exist sequences of random vectors $\{\mathbf{N}^*(l) : l \geq 1\}$ and $\{\mathbf{I}^*(l) : l \geq 1\}$ such that*

$$\mathcal{L}(\mathbf{I}^*(l)) = P_l \text{ and } \mathcal{L}(\mathbf{N}^*(l)) = Q_l \text{ for every } l, \text{ and } \|\mathbf{I}^*(l) - \mathbf{N}^*(l)\| \rightarrow 0, \text{ a.s.-}\mathbb{P}^*, \quad (6)$$

where $\mathcal{L}(\cdot)$ indicates the law of a given random vector, and $\|\cdot\|$ is the Euclidean norm.

3 Main results

Fix integers $k \geq 1$ and $d_1, \dots, d_k \geq 1$, and consider a sequence of k -dimensional random vectors of the type

$$\mathbf{I}(l) = \left(I_{d_1} \left(f_l^{(1)} \right), \dots, I_{d_k} \left(f_l^{(k)} \right) \right), \quad l \geq 1, \quad (7)$$

where, for each $l \geq 1$ and every $j = 1, \dots, k$, $f_l^{(j)}$ is an element of $\mathfrak{H}^{\odot d_j}$. We will suppose the following:

- There exists $\eta > 0$ such that $\left\| f_l^{(j)} \right\|_{\mathfrak{H}^{\odot d_j}} \geq \eta$, for every $j = 1, \dots, k$ and every $l \geq 1$.
- For every $j = 1, \dots, k$, the sequence

$$\mathbb{E} \left[I_{d_j} \left(f_l^{(j)} \right)^2 \right] = d_j! \left\| f_l^{(j)} \right\|_{\mathfrak{H}^{\odot d_j}}^2, \quad l \geq 1, \quad (8)$$

is bounded.

Note that the integers d_1, \dots, d_k do not depend on l . For every $l \geq 1$, we denote by $\mathbf{N}(l) = \left(N_l^{(1)}, \dots, N_l^{(k)} \right)$ a centered k -dimensional Gaussian vector with the same covariance matrix as $\mathbf{I}(l)$, that is,

$$\mathbb{E} \left[N_l^{(i)} N_l^{(j)} \right] = \mathbb{E} \left[I_{d_i} \left(f_l^{(i)} \right) I_{d_j} \left(f_l^{(j)} \right) \right], \quad (9)$$

for every $1 \leq i, j \leq k$. For every $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$, we also use the compact notation: $\langle \lambda, \mathbf{I}(l) \rangle_k = \sum_{j=1}^k \lambda_j I_{d_j} \left(f_l^{(j)} \right)$ and $\langle \lambda, \mathbf{N}(l) \rangle_k = \sum_{j=1}^k \lambda_j N_l^{(j)}$.

The next result is one of the main contributions of this paper. Its proof is deferred to Section 4.

Theorem 1 *Let the above notation and assumptions prevail, and suppose that, for every $j = 1, \dots, k$, the following asymptotic condition holds: for every $p = 1, \dots, d_j - 1$,*

$$\left\| f_l^{(j)} \otimes_p f_l^{(j)} \right\|_{\mathfrak{H}^{\odot 2(d_j-p)}} \rightarrow 0, \quad \text{as } l \rightarrow +\infty. \quad (10)$$

Then, as $l \rightarrow +\infty$ and for every compact set $M \subset \mathbb{R}^k$,

$$\sup_{\lambda \in M} |\mathbb{E} [\exp(i \langle \lambda, \mathbf{I}(l) \rangle_k)] - \mathbb{E} [\exp(i \langle \lambda, \mathbf{N}(l) \rangle_k)]| \rightarrow 0. \quad (11)$$

We now state two crucial consequences of Theorem 1. The first one (Proposition 2) provides a formal meaning to the intuitive fact that, since (11) holds and since the variances of $\mathbf{I}(l)$ do not explode, the laws of $\mathbf{I}(l)$ and $\mathbf{N}(l)$ are “asymptotically close”. The second one (Theorem 3) combines Theorem 1 and Proposition 2 to obtain an exhaustive generalization “without covariance conditions” of Theorem 0 (see the Introduction). Note that in the statement of Theorem 3 also appear Malliavin operators, so that our results are a genuine extension of the main findings by Nualart and Ortiz-Latorre in [13]. We stress that multiple stochastic integrals of the type $I_d(f)$, $d \geq 1$ and $f \in \mathfrak{H}^{\odot d}$, are always such that $I_d(f) \in \mathbb{D}^{1,2}$.

Proposition 2 *Let the assumptions of Theorem 1 prevail (in particular, (10) holds), and denote by $\mathcal{L}(\mathbf{I}(l))$ and $\mathcal{L}(\mathbf{N}(l))$, respectively, the law of $\mathbf{I}(l)$ and $\mathbf{N}(l)$, $l \geq 1$. Then, the two collections $\{\mathcal{L}(\mathbf{N}(l)) : l \geq 1\}$ and $\{\mathcal{L}(\mathbf{I}(l)) : l \geq 1\}$ are tight. Moreover, if $\gamma(\cdot, \cdot)$ metrizes the weak convergence on $\mathbf{P}(\mathbb{R}^k)$, then*

$$\lim_{l \rightarrow +\infty} \gamma(\mathcal{L}(\mathbf{I}(l)), \mathcal{L}(\mathbf{N}(l))) = 0. \quad (12)$$

Proof. The fact that $\{\mathcal{L}(\mathbf{N}(l)) : l \geq 1\}$ and $\{\mathcal{L}(\mathbf{I}(l)) : l \geq 1\}$ are tight is a consequence of the boundedness of the sequence (8) and of the relation $\mathbb{E}[I_{d_j}(f_l^{(j)})^2] = \mathbb{E}[(N_l^{(j)})^2]$. The rest of the proof is standard, and is provided for the sake of completeness. We shall prove (12) by contradiction. Suppose there exist $\varepsilon > 0$ and a subsequence $\{l_n\}$ such that $\gamma(\mathcal{L}(\mathbf{I}(l_n)), \mathcal{L}(\mathbf{N}(l_n))) > \varepsilon$ for every n . Tightness implies that $\{l_n\}$ must contain a subsequence $\{l_{n'}\}$ such that $\mathcal{L}(\mathbf{I}(l_{n'}))$ and $\mathcal{L}(\mathbf{N}(l_{n'}))$ are both weakly convergent. Since (11) holds, we deduce that $\mathcal{L}(\mathbf{I}(l_{n'}))$ and $\mathcal{L}(\mathbf{N}(l_{n'}))$ must necessarily converge to the same weak limit, say Q . The fact that γ metrizes the weak convergence implies finally that

$$\gamma(\mathcal{L}(\mathbf{I}(l_{n'})), \mathcal{L}(\mathbf{N}(l_{n'}))) \leq \gamma(\mathcal{L}(\mathbf{I}(l_{n'})), Q) + \gamma(\mathcal{L}(\mathbf{N}(l_{n'})), Q) \xrightarrow{n' \rightarrow +\infty} 0, \quad (13)$$

thus contradicting the former assumptions on $\{l_n\}$ (note that the inequality in (13) is just the triangle inequality). This shows that (12) must necessarily take place. ■

Remarks. (i) A result analogous to the arguments used in the proof of Corollary 2 is stated in [4, Exercise 3, p. 419]. Note also that, without tightness, a condition such as (11) does *not* allow to deduce the asymptotic relation (12). See for instance [4, Proposition 11.7.6] for a counterexample involving the Prokhorov metric on $\mathbf{P}(\mathbb{R})$.

(ii) Since (12) holds in particular when γ is equal to the Prokhorov metric or the Fortet-Mounier metric (as defined in (4) and (5)), Proposition 2 implies that, on some auxiliary probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$, there exist sequences of random vectors $\{\mathbf{N}^*(l) : l \geq 1\}$ and $\{\mathbf{I}^*(l) : l \geq 1\}$ such that

$$\mathbf{I}^*(l) \stackrel{\text{law}}{=} \mathbf{I}(l) \text{ and } \mathbf{N}^*(l) \stackrel{\text{law}}{=} \mathbf{N}(l) \text{ for every } l, \text{ and } \|\mathbf{I}^*(l) - \mathbf{N}^*(l)\| \rightarrow 0, \text{ a.s.-}\mathbb{P}^*, \quad (14)$$

where $\|\cdot\|$ stands for the Euclidean norm (see (6), as well as [4, Theorem 11.7.1]).

Theorem 3 *Suppose that the sequence $\mathbf{I}(l)$, $l \geq 1$, verifies the assumptions of this section (in particular, for every $j = 1, \dots, k$, the sequence of variances appearing in (8) is bounded). Then, the following conditions are equivalent.*

1. *As $l \rightarrow +\infty$, relation (10) is satisfied for every $j = 1, \dots, k$ and every $p = 1, \dots, d_j - 1$;*
- 2.

$$\lim_{l \rightarrow +\infty} \rho(\mathcal{L}(\mathbf{I}(l)), \mathcal{L}(\mathbf{N}(l))) = \lim_{l \rightarrow +\infty} \beta(\mathcal{L}(\mathbf{I}(l)), \mathcal{L}(\mathbf{N}(l))) = 0 \quad (15)$$

where ρ and β are, respectively, the Prokhorov metric and the Fortet-Mounier metric, as defined in (4) and (5);

3. *As $l \rightarrow +\infty$, for every $j = 1, \dots, k$,*

$$\mathbb{E} \left[I_{d_j} \left(f_l^{(j)} \right)^4 \right] - 3 \mathbb{E} \left[I_{d_j} \left(f_l^{(j)} \right)^2 \right]^2 = \mathbb{E} \left[I_{d_j} \left(f_l^{(j)} \right)^4 \right] - 3(d_j!)^2 \left\| f_l^{(j)} \right\|_{\mathfrak{H}^{\odot d_j}}^4 \rightarrow 0;$$

4. For every $j = 1, \dots, k$,

$$\lim_{l \rightarrow +\infty} \rho \left(\mathcal{L} \left(I_{d_j} \left(f_l^{(j)} \right) \right), \mathcal{L} \left(N_l^{(j)} \right) \right) = \lim_{l \rightarrow +\infty} \beta \left(\mathcal{L} \left(I_{d_j} \left(f_l^{(j)} \right) \right), \mathcal{L} \left(N_l^{(j)} \right) \right) = 0, \quad (16)$$

where ρ and β are the Prokhorov and Fortet-Mounier metric on \mathbb{R} ;

5. For every $j = 1, \dots, k$,

$$\left\| D \left[I_{d_j} \left(f_l^{(j)} \right) \right] \right\|_{\mathfrak{H}}^2 - d_j (d_j!) \left\| f_l^{(j)} \right\|_{\mathfrak{H}^{\otimes d_j}}^2 \rightarrow 0, \quad \text{in } L^2(X), \quad (17)$$

as $l \rightarrow +\infty$, where D is the Malliavin derivative operator defined in Section 2.

Proof. The implication 1. \implies 2., is a consequence of Theorem 1 and Proposition 2. Now suppose (15) is in order. Then, according to [4, Theorem 11.7.1], on a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$, there exist sequences of random vectors $\mathbf{N}^*(l) = (N_l^{*,(1)}, \dots, N_l^{*,(j)})$, $l \geq 1$, and $\mathbf{I}^*(l) = (I_l^{*,(1)}, \dots, I_l^{*,(k)})$, $l \geq 1$, such that (14) takes place. Now

$$3\mathbb{E} \left[I_{d_j} \left(f_l^{(j)} \right)^2 \right]^2 = 3\mathbb{E} \left[\left(N_l^{(j)} \right)^2 \right]^2 = \mathbb{E} \left[\left(N_l^{(j)} \right)^4 \right] = \mathbb{E}^* \left[\left(N_l^{*,(j)} \right)^4 \right],$$

for every $j = 1, \dots, k$, so that

$$\mathbb{E} \left[I_{d_j} \left(f_l^{(j)} \right)^4 \right] - 3\mathbb{E} \left[I_{d_j} \left(f_l^{(j)} \right)^2 \right]^2 = \mathbb{E}^* \left[\left(I_l^{*,(j)} \right)^4 - \left(N_l^{*,(j)} \right)^4 \right] \xrightarrow{l \rightarrow +\infty} 0. \quad (18)$$

The convergence to zero in (18) is a consequence of the boundedness of the sequence (8), implying that the family $A_l^* = \left(I_l^{*,(j)} \right)^4 - \left(N_l^{*,(j)} \right)^4$, $l \geq 1$, is uniformly integrable. To see why $\{A_l^*\}$ is uniformly integrable, one can use the fact that, since each $I_l^{*,(j)}$ has the same law as an element of the d_j th chaos of X and each $N_l^{*,(j)}$ is Gaussian, then (see e.g. [6, Ch. VI]) for every $p \geq 2$ there exists a universal positive constant $C_{p,j}$ (independent of l) such that

$$\begin{aligned} \mathbb{E} [|A_l^*|^p]^{1/p} &= \mathbb{E}^* \left[\left| \left(I_l^{*,(j)} \right)^4 - \left(N_l^{*,(j)} \right)^4 \right|^p \right]^{1/p} \\ &\leq \mathbb{E}^* \left[\left(I_l^{*,(j)} \right)^{4p} \right]^{1/4p} + \mathbb{E}^* \left[\left(N_l^{*,(j)} \right)^{4p} \right]^{1/4p} \\ &\leq C_{p,j} \mathbb{E}^* \left[\left(I_l^{*,(j)} \right)^2 \right]^2 + C_{p,j} \mathbb{E}^* \left[\left(N_l^{*,(j)} \right)^2 \right]^2 \\ &= 2C_{p,j} \times (d_j!)^2 \left\| f_l^{(j)} \right\|_{\mathfrak{H}^{\otimes d_j}}^4 \leq 2C_{p,j} M_j, \end{aligned}$$

where $M_j = \sup_l (d_j!)^2 \left\| f_l^{(j)} \right\|^4 < +\infty$, due to (8). This proves that 2. \implies 3.. The implication 3. \implies 1. can be deduced from the formula (proved in [14, p. 183])

$$\begin{aligned} &\mathbb{E} \left[I_{d_j} \left(f_l^{(j)} \right)^4 \right] - 3\mathbb{E} \left[I_{d_j} \left(f_l^{(j)} \right)^2 \right]^2 = \mathbb{E} \left[I_{d_j} \left(f_l^{(j)} \right)^4 \right] - 3(d_j!)^2 \left\| f_l^{(j)} \right\|_{\mathfrak{H}^{\otimes d_j}}^4 \\ &= \sum_{p=1}^{d_j-1} \frac{(d_j!)^4}{(p! (d_j - p)!)^2} \left\{ \left\| f_l^{(j)} \otimes_p f_l^{(j)} \right\|_{\mathfrak{H}^{\otimes 2(d_j-p)}}^2 + \binom{2(d_j-p)}{d_j-p} \left\| f_l^{(j)} \tilde{\otimes}_p f_l^{(j)} \right\|_{\mathfrak{H}^{\otimes 2(d_j-p)}}^2 \right\}, \end{aligned}$$

The equivalence 1. \iff 4. is an immediate consequence of the previous discussion.

To conclude the proof, we shall now show the double implication 1. \iff 5.. To do this, we first observe that, by performing the same calculations as in [13, Proof of Lemma 2] (which are based on an application of the multiplication formulae for multiple integrals, see [12, Proposition 1.1.3]), one obtains that

$$\begin{aligned} \left\| D \left[I_{d_j} \left(f_l^{(j)} \right) \right] \right\|_{\mathfrak{H}}^2 &= d_j (d_j!) \left\| f_l^{(j)} \right\|_{\mathfrak{H}^{\otimes d_j}}^2 \\ &\quad + d_j^2 \sum_{p=1}^{d_j-1} (p-1)! \binom{n-1}{p-1}^2 I_{2(d_j-p)} \left(f_l^{(j)} \widetilde{\otimes}_p f_l^{(j)} \right). \end{aligned}$$

Since $\left\| f_l^{(j)} \otimes_p f_l^{(j)} \right\|_{\mathfrak{H}^{\otimes 2(d_j-p)}}^2 \geq \left\| f_l^{(j)} \widetilde{\otimes}_p f_l^{(j)} \right\|_{\mathfrak{H}^{\otimes 2(d_j-p)}}^2$, the last relation implies immediately that 1. \Rightarrow 5.. To prove the opposite implication, first observe that, due to the boundedness of (8) and the Cauchy-Schwarz inequality, there exists a finite constant M (independent of j and l) such that

$$\left\| f_l^{(j)} \otimes_p f_l^{(j)} \right\|_{\mathfrak{H}^{\otimes 2(d_j-p)}}^2 \leq \left\| f_l^{(j)} \right\|_{\mathfrak{H}^{\otimes d_j}}^4 \leq M.$$

This implies that, for every sequence $\{l_n\}$, there exists a subsequence $\{l_{n'}\}$ such that the sequences $\left\| f_{l_{n'}}^{(j)} \otimes_p f_{l_{n'}}^{(j)} \right\|_{\mathfrak{H}^{\otimes 2(d_j-p)}}^2$ and $d_j! \left\| f_{l_{n'}}^{(j)} \right\|_{\mathfrak{H}^{\otimes d_j}}^2$ are convergent for every $j = 1, \dots, k$ and every $p = 1, \dots, d_j - 1$ (recall that, by assumption, there exists a constant $\eta > 0$, such that $\left\| f_{l_{n'}}^{(j)} \right\|_{\mathfrak{H}^{\otimes d_j}} \geq \eta$, for every j and l). We shall now prove that, whenever (17) is verified, then necessarily $\left\| f_{l_{n'}}^{(j)} \otimes_p f_{l_{n'}}^{(j)} \right\|_{\mathfrak{H}^{\otimes 2(d_j-p)}}^2 \rightarrow 0$. Indeed, Theorem 4 in [13] implies that, if (17) takes place and $d_j! \left\| f_{l_{n'}}^{(j)} \right\|_{\mathfrak{H}^{\otimes d_j}}^2 \rightarrow c > 0$, then necessarily

$$I_{d_j} \left(f_{l_{n'}}^{(j)} \right) \xrightarrow{Law} N(0, c), \quad (19)$$

where $N(0, c)$ stands for a centered Gaussian random variable with variance c . But Theorem 1 in [14] implies that, if (19) is verified, then $\left\| f_{l_{n'}}^{(j)} \otimes_p f_{l_{n'}}^{(j)} \right\|_{\mathfrak{H}^{\otimes 2(d_j-p)}}^2 \rightarrow 0$, thus proving our claim. This shows that 5. \Rightarrow 1.. ■

The next result says that, under the additional assumption that the variances of the elements of $\mathbf{I}(l)$ converge to one, the asymptotic approximation (15) is equivalent to the fact that each component of $\mathbf{I}(l)$ verifies a CLT. The proof is elementary, and therefore omitted.

Corollary 4 *Fix $k \geq 2$, and suppose that the sequence $\mathbf{I}(l)$, $l \geq 1$, is such that, for every $j = 1, \dots, k$, the sequence of variances appearing in (8) converges to 1, as $l \rightarrow +\infty$. Then, each one of Conditions 1.-5. in the statement of Theorem 3 is equivalent to the following: for every $j = 1, \dots, k$,*

$$I_{d_j} \left(f_l^{(j)} \right) \xrightarrow[l \rightarrow +\infty]{Law} N(0, 1), \quad (20)$$

where $N(0, 1)$ is a centered Gaussian random variable with unitary variance.

Remark. The results of this section can be suitably extended to deal with the Gaussian approximations of random vectors of the type $(F_l^{(1)}(X), \dots, F_l^{(k)}(X))$, where $F_l^{(j)}(X)$, $j = 1, \dots, k$, is a general square integrable functional of the isonormal process X , not necessarily having the form of a multiple integral. See [10, Th. 6] for a statement containing an extension of this type.

4 Proof of Theorem 1

We provide the proof in the case where

$$\mathfrak{H} = L^2([0, 1], \mathcal{B}([0, 1]), dx) = L^2([0, 1]), \quad (21)$$

where dx stands for Lebesgue measure. The extension to a general \mathfrak{H} is obtained by using the same arguments outlined in [14, Section 2.2]. If \mathfrak{H} is as in (21), then for every $d \geq 2$ one has that $\mathfrak{H}^{\odot d} = L_s^2([0, 1]^d)$, where the symbol $L_s^2([0, 1]^d)$ indicates the class of symmetric, real-valued and square-integrable functions (with respect to the Lebesgue measure) on $[0, 1]^d$. Also, the isonormal process X coincides with the Gaussian space generated by the standard Brownian motion

$$t \mapsto W_t \triangleq X(1_{[0, t]}), \quad t \in [0, 1].$$

This implies in particular that, for every $d \geq 2$, the Wiener-Itô integral $I_d(f)$, $f \in L_s^2([0, 1]^d)$, can be rewritten in terms of an iterated stochastic integral with respect to W , that is:

$$I_d(f) = d! \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{d-1}} f(t_1, \dots, t_d) dW_{t_d} \cdots dW_{t_2} dW_{t_1}. \quad (22)$$

We also have that $I_1(f) = \int_0^1 f(s) dW_s$ for every $f \in L_s^2([0, 1]^1) \equiv L^2([0, 1])$. Note that the RHS of (22) is just an iterated adapted stochastic integral of the Itô type. Finally, for every $f \in L_s^2([0, 1]^d)$, every $g \in L_s^2([0, 1]^{d'})$ and every $p = 0, \dots, d \wedge d'$, we observe that the contraction $f \otimes_p g$ is the (not necessarily symmetric) element of $L^2([0, 1]^{d+d'-2p})$ given by:

$$\begin{aligned} f \otimes_p g(y_1, \dots, y_{d+d'-2p}) &= \int_{[0, 1]^p} f(y_1, \dots, y_{d-p}, a_1, \dots, a_p) \times \\ &\quad \times g(y_{d-p+1}, \dots, y_{d+d'-2p}, a_1, \dots, a_p) da_1 \dots da_p. \end{aligned} \quad (23)$$

In the framework of (21), the proof of Theorem 1 relies on some computations contained in [15], as well as on an appropriate use of the *Burkholder-Davis-Gundy inequalities* (see for instance [16, Ch. IV §4]). Fix $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$, and consider the random variable

$$\begin{aligned} \langle \lambda, \mathbf{I}(l) \rangle_k &= \sum_{j=1}^k \lambda_j d_j! \int_0^1 \cdots \int_0^{u_{d_j-1}} f_l^{(j)}(u_1, \dots, u_{d_j}) dW_{u_{d_j}} \cdots dW_{u_1} \\ &\triangleq \sum_{j=1}^k \lambda_j d_j! J_{d_j}^1(f_l^{(j)}) = \int_0^1 \left(\sum_{j=1}^k \lambda_j d_j! J_{d_j-1}^u(f_l^{(j)}(u, \cdot)) \right) dW_u \\ &= \int_0^1 \left(\sum_{j=1}^k \lambda_j d_j I_{d_j-1}(f_l^{(j)}(u, \cdot) \mathbf{1}_{[0, u]^{d_j-1}}) \right) dW_u, \end{aligned}$$

where, for every $d \geq 1$, every $t \in [0, 1]$ and every $f \in L_s^2([0, 1]^d)$, we define $J_d^t(f) = I_d(f \mathbf{1}_{[0, t]^d}) / d!$ (for every $c \in \mathbb{R}$, we also use the conventional notation $J_0^t(c) = c$). We start by recalling some preliminary results involving Brownian martingales. Start by setting, for every $u \in [0, 1]$, $\phi_{\lambda, l}(u) = \sum_{j=1}^k \lambda_j d_j I_{d_j-1}(f_l^{(j)}(u, \cdot) \mathbf{1}_{[0, u]^{d_j-1}})$, and observe that the random application

$$t \mapsto \sum_{j=1}^k \lambda_j d_j! J_{d_j}^t(f_l^{(j)}) = \int_0^t \phi_{\lambda, l}(u) dW_u, \quad t \in [0, 1],$$

defines a (continuous) square-integrable martingale started from zero, with respect to the canonical filtration of W , noted $\{\mathcal{F}_t^W : t \in [0, 1]\}$. The quadratic variation of this martingale is classically given by $t \mapsto \int_0^t \phi_{\lambda, l}(u)^2 du$, and a standard application of the Dambis, Dubins and Schwarz Theorem (see [16, Ch. V §1]) yields that, for every $l \geq 1$, there exists a standard Brownian motion (initialized at zero) $W^{(\lambda, l)} = \{W_t^{(\lambda, l)} : t \geq 0\}$ such that

$$\langle \lambda, \mathbf{I}(l) \rangle_k = \int_0^1 \phi_{\lambda, l}(u) dW_u = W_{\int_0^1 \phi_{\lambda, l}(u)^2 du}^{(\lambda, l)}.$$

Note that, in general, the definition of $W^{(\lambda, l)}$ strongly depends on λ and l , and that $W^{(\lambda, l)}$ is *not* a \mathcal{F}_t^W -Brownian motion. However, the following relation links the two Brownian motions $W^{(\lambda, l)}$ and W : there exists a (continuous) filtration $\{\mathcal{G}_t^{(\lambda, l)} : t \geq 0\}$ such that (i) $W_t^{(\lambda, l)}$ is a $\mathcal{G}_t^{(\lambda, l)}$ -Brownian motion, and (ii) for every fixed $s \in [0, 1]$ the positive random variable $\int_0^s \phi_{\lambda, l}(u)^2 du$ is a $\mathcal{G}_t^{(\lambda, l)}$ -stopping time. Now define the positive constant (which is trivially a $\mathcal{G}_t^{(\lambda, l)}$ -stopping time)

$$q(\lambda, l) = \int_0^1 \mathbb{E}(\phi_{\lambda, l}(u)^2) du,$$

and observe that the usual properties of complex exponentials and a standard application of the Burkholder-Davis-Gundy inequality (in the version stated in [16, Corollary 4.2, Ch. IV]) yield the following estimates:

$$\begin{aligned} \left| \mathbb{E}[\exp(i \langle \lambda, \mathbf{I}(l) \rangle_k)] - \mathbb{E}[\exp(i W_{q(\lambda, l)}^{(\lambda, l)})] \right| &= \left| \mathbb{E}\left[\exp\left(i W_{\int_0^1 \phi_{\lambda, l}(u)^2 du}^{(\lambda, l)}\right)\right] - \mathbb{E}\left[\exp\left(i W_{q(\lambda, l)}^{(\lambda, l)}\right)\right] \right| \\ &\leq \mathbb{E}\left[\left|W_{\int_0^1 \phi_{\lambda, l}(u)^2 du}^{(\lambda, l)} - W_{q(\lambda, l)}^{(\lambda, l)}\right|\right] \\ &\leq \mathbb{E}\left[\left|W_{\int_0^1 \phi_{\lambda, l}(u)^2 du}^{(\lambda, l)} - W_{q(\lambda, l)}^{(\lambda, l)}\right|^4\right]^{\frac{1}{4}} \\ &\leq C \mathbb{E}\left[\left|\int_0^1 \phi_{\lambda, l}(u)^2 du - q(\lambda, l)\right|^2\right]^{\frac{1}{4}}, \end{aligned} \quad (24)$$

where C is some universal constant independent of λ and l . To see how to obtain the inequality (24), introduce first the shorthand notation $T(\lambda, l) \triangleq \int_0^1 \phi_{\lambda, l}(u)^2 du$ (recall that $T(\lambda, l)$ is a $\mathcal{G}_t^{(\lambda, l)}$ -stopping time), and then write

$$\left|W_{\int_0^1 \phi_{\lambda, l}(u)^2 du}^{(\lambda, l)} - W_{q(\lambda, l)}^{(\lambda, l)}\right| = \left|\int_{T(\lambda, l) \wedge q(\lambda, l)}^{T(\lambda, l) \vee q(\lambda, l)} dW_u^{(\lambda, l)}\right| = \left|\int_0^{T(\lambda, l) \vee q(\lambda, l)} H(u) dW_u^{(\lambda, l)}\right|,$$

where $H(u)$ is the $\mathcal{G}_u^{(\lambda,l)}$ -predictable process given by $H(u) = \mathbf{1}\{u \geq T(\lambda, l) \wedge q(\lambda, l)\}$, so that

$$\begin{aligned} \left| \int_0^{T(\lambda,l) \vee q(\lambda,l)} H(u)^2 du \right| &= |T(\lambda, l) \wedge q(\lambda, l) - T(\lambda, l) \vee q(\lambda, l)| \\ &= |T(\lambda, l) - q(\lambda, l)| = \left| \int_0^1 \phi_{\lambda,l}(u)^2 du - q(\lambda, l) \right|. \end{aligned}$$

In particular, relation (24) yields that the proof of Theorem 1 is concluded, once the following two facts are proved: (A) $W_{q(\lambda,l)}^{(\lambda,l)} = \langle \lambda, \mathbf{N}(l) \rangle_k$, for every $\lambda \in \mathbb{R}^k$ and every $l \geq 1$; (B) the sequence

$$\mathbb{E} \left[\left| \int_0^1 \phi_{\lambda,l}(u)^2 du - q(\lambda, l) \right|^2 \right], \quad l \geq 1,$$

converges to zero, uniformly in λ , on every compact set of the type $M = [-T, T]^k$, where $T \in (0, +\infty)$. The proof of (A) is immediate: indeed, $W^{(\lambda,l)}$ is a standard Brownian motion and, by using the isometric properties of stochastic integrals and the fact that the covariance structures of $\mathbf{N}(l)$ and $\mathbf{I}(l)$ coincide,

$$q(\lambda, l) = \int_0^1 \mathbb{E}(\phi_{\lambda,l}(u)^2) du = \mathbb{E} \left[\left(\int_0^1 \phi_{\lambda,l}(u) dW_u \right)^2 \right] = \mathbb{E} \left[\langle \lambda, \mathbf{I}(l) \rangle_k^2 \right] = \mathbb{E} \left[\langle \lambda, \mathbf{N}(l) \rangle_k^2 \right].$$

To prove (B), use a standard version of the multiplication formula between multiple stochastic integrals (see for instance [12, Proposition 1.5.1])

$$\begin{aligned} \int_0^1 \phi_{\lambda,l}(u)^2 du &= \int_0^1 \left(\sum_{j=1}^k \lambda_j d_j I_{d_j-1} \left(f_l^{(j)}(u, \cdot) \mathbf{1}_{[0,u]^{d_j-1}} \right) \right)^2 du \\ &= \int_0^1 \sum_{j,i=1}^k \lambda_j \lambda_i d_j d_i I_{d_i-1} \left(f_l^{(i)}(u, \cdot) \mathbf{1}_{[0,u]^{d_i-1}} \right) I_{d_j-1} \left(f_l^{(j)}(u, \cdot) \mathbf{1}_{[0,u]^{d_j-1}} \right) du \\ &= q(\lambda, l) + \sum_{j,i=1}^k \lambda_j \lambda_i d_j d_i \int_0^1 \sum_{p=0}^{D(i,j)} \binom{d_i-1}{p} \binom{d_j-1}{p} \\ &\quad \times I_{d_i+d_j-2-2p} \left((f_l^{(i)}(u, \cdot) \mathbf{1}_{[0,u]^{d_i-1}}) \otimes_p (f_l^{(j)}(u, \cdot) \mathbf{1}_{[0,u]^{d_j-1}}) \right), \end{aligned} \tag{25}$$

where the index $D(i, j)$ is defined as

$$D(i, j) = \begin{cases} d_i - 2 & \text{if } d_i = d_j \\ \min(d_i, d_j) - 1 & \text{if } d_i \neq d_j. \end{cases}$$

Formula (25) implies that, for every $\lambda \in [-T, T]^k$ ($T > 0$),

$$\begin{aligned} & \mathbb{E} \left[\left| \int_0^1 \phi_{\lambda, l}(u)^2 du - q(\lambda, l) \right|^2 \right]^{\frac{1}{2}} \\ & \leq (T \max_i d_i)^2 \sum_{i, j=1}^k \sum_{p=0}^{D(i, j)} \binom{d_i - 1}{p} \binom{d_j - 1}{p} \\ & \quad \times \mathbb{E} \left[\left(\int_0^1 I_{d_i + d_j - 2 - 2p} \left((f_l^{(i)}(u, \cdot) \mathbf{1}_{[0, u]^{d_i - 1}}) \otimes_p (f_l^{(j)}(u, \cdot) \mathbf{1}_{[0, u]^{d_j - 1}}) \right) du \right)^2 \right]^{\frac{1}{2}} \end{aligned} \quad (26)$$

(note that the RHS of (26) does not depend on λ). Finally, a direct application of the calculations contained in [15, p. 253-255] yields that, for every $i, j = 1, \dots, k$ and every $p = 0, \dots, D(i, j)$,

$$\mathbb{E} \left[\left(\int_0^1 I_{d_i + d_j - 2 - 2p} \left((f_l^{(i)}(u, \cdot) \mathbf{1}_{[0, u]^{d_i - 1}}) \otimes_p (f_l^{(j)}(u, \cdot) \mathbf{1}_{[0, u]^{d_j - 1}}) \right) du \right)^2 \right]^{\frac{1}{2}} \rightarrow 0, \quad (27)$$

as $l \rightarrow +\infty$. This concludes the proof of Theorem 1. ■

Remark. By inspection of the calculations contained in [15, p. 253-255], it is easily seen that, to deduce (27) from (10), it is necessary that the sequence of variances (8) is bounded.

5 Concluding remarks on applications

Theorem 1 and Theorem 3 are used in [10] to deduce high-frequency asymptotic results for subordinated spherical random fields. This study is strongly motivated by the probabilistic modelling and statistical analysis of the Cosmic Microwave Background radiation (see [7], [8], [9] and [10] for a detailed discussion of these applications). In what follows, we provide a brief presentation of some of the results obtained in [10].

Let $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ be the unit sphere, and let $T = \{T(x) : x \in \mathbb{S}^2\}$ be a real-valued (centered) Gaussian field which is also *isotropic*, in the sense that $T(x) \stackrel{Law}{=} T(\mathcal{R}x)$ (in the sense of stochastic processes) for every rotation $\mathcal{R} \in SO(3)$. The following facts are well known:

- (1) The trajectories of T admit the harmonic expansion $T(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^l a_{lm} Y_{lm}(x)$, where $\{Y_{lm} : l \geq 0, m = -l, \dots, l\}$ is the class of *spherical harmonics* (defined e.g. in [17, Ch. 5]);
- (2) The complex-valued array of harmonic coefficients $\{a_{lm} : l \geq 0, m = -l, \dots, l\}$ is composed of centered Gaussian random variables such that the variances $\mathbb{E}|a_{lm}|^2 \triangleq C_l$ depend exclusively on l (see for instance [1]);
- (3) The law of T is completely determined by the *power spectrum* $\{C_l : l \geq 0\}$ defined at the previous point.

Now fix $q \geq 2$, and consider the subordinated field

$$T^{(q)}(x) \triangleq H_q(T(x)), \quad x \in \mathbb{S}^2,$$

where H_q is the q th Hermite polynomial. Plainly, the field $T^{(q)}$ is isotropic and admits the harmonic expansion

$$T^{(q)}(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^l a_{lm;q} Y_{lm}(x) \triangleq \sum_{l=0}^{\infty} T_l^{(q)}(x),$$

where $a_{lm;q} \triangleq \int_{\mathbb{S}^2} T^{(q)}(z) \overline{Y_{lm}(z)} dz$. For every $l \geq 0$, the field $T_l^{(q)} = \sum_{m=-l}^l a_{lm;q} Y_{lm}$ is real-valued and isotropic, and it is called the l th frequency component of $T^{(q)}$ (see [7] or [10] for a physical interpretation of frequency components). In [10], the following problem is studied.

Problem A. Fix $q \geq 2$. Find conditions on the power spectrum $\{C_l : l \geq 0\}$ to have that the finite dimensional distributions (f.d.d.'s) of the normalized frequency field

$$\overline{T}_l^{(q)}(x) \triangleq \frac{T_l^{(q)}(x)}{\text{Var}(T_l^{(q)}(x))^{1/2}}, \quad x \in \mathbb{S}^2,$$

are ‘asymptotically close to Gaussian’, as $l \rightarrow +\infty$.

The main difficulty when dealing with Problem A is that (due to isotropy) one has always that

$$\mathbb{E} [\overline{T}_l^{(q)}(x) \overline{T}_l^{(q)}(y)] = P_l(\cos \langle x, y \rangle), \quad (28)$$

where P_l is the l th Legendre polynomial, and $\langle x, y \rangle$ is the angle between x and y . Indeed, since in general the quantity $P_l(\cos \langle x, y \rangle)$ does not converge (as $l \rightarrow +\infty$), one cannot prove that the f.d.d.'s of $\overline{T}_l^{(q)}$ converge to those of a Gaussian field (even if $\overline{T}_l^{(q)}(x)$ converges in law to a Gaussian random variable for every fixed x). However, as an application of Theorem 1 and Proposition 2, one can prove the following approximation result.

Proposition 5 Under the above notation and assumptions, suppose that, for any fixed $x \in \mathbb{S}^2$,

$$\overline{T}_l^{(q)}(x) \xrightarrow[l \rightarrow +\infty]{Law} N(0, 1). \quad (29)$$

Then, for any $k \geq 1$, any $x_1, \dots, x_k \in \mathbb{S}^2$ and any γ metrizing the weak convergence on $\mathbf{P}(\mathbb{R}^k)$,

$$\gamma \left(\mathcal{L} \left(\overline{T}_l^{(q)}(x_1), \dots, \overline{T}_l^{(q)}(x_k) \right), \mathbf{N}(l) \right) \xrightarrow[l \rightarrow +\infty]{} 0, \quad (30)$$

where, for every l , $\mathbf{N}(l) = (N_l^{(1)}, \dots, N_l^{(k)})$ is a centered real-valued Gaussian vector such that

$$\mathbb{E} \left\{ N_l^{(i)} N_l^{(j)} \right\} = P_l(\cos \langle x_i, x_j \rangle).$$

Proof. Since $\overline{T}_l^{(q)}(x)$ is a linear functional involving uniquely Hermite polynomials of order q (written on the Gaussian field T) one deduces that there exists a real Hilbert space \mathfrak{H} such that (in the sense of stochastic processes)

$$\overline{T}_l^{(q)}(x) \stackrel{Law}{=} I_q(f_{(q,l,x)}),$$

where the class of symmetric kernels

$$\{f_{(q,l,x)} : l \geq 0, x \in \mathbb{S}^2\}$$

is a subset of $\mathfrak{H}^{\odot q}$, and $I_q(f_{(q,l,x)})$ stands for the q th Wiener-Itô integral of $f_{(q,l,x)}$ with respect to an isonormal Gaussian process over \mathfrak{H} , as defined in Section 2. Since the variances of the components of the vector $(\overline{T}_l^{(q)}(x_1), \dots, \overline{T}_l^{(q)}(x_k))$ are all equal to 1 by construction, we can apply Theorem 3 and Proposition 2. Indeed, by Theorem 3 we know that (29) implies that, for every $p = 1, \dots, q-1$ and every $j = 1, \dots, k$,

$$f_{(q,l,x_j)} \otimes_p f_{(q,l,x_j)} \rightarrow 0 \text{ in } \mathfrak{H}^{\odot 2(q-p)}.$$

Finally, Proposition 2 and (28) imply immediately the desired conclusion. ■

The derivation of sufficient conditions to have (29) is the main object of [10]. In particular, it is proved that sufficient (and sometimes also necessary) conditions for (29) can be neatly expressed in terms of the so-called *Clebsch-Gordan coefficients* (see again [17]), that are elements of unitary matrices connecting reducible representations of $SO(3)$.

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